

A NOTE ON DEGENERATE APOSTOL-BERNOULLI NUMBERS AND POLYNOMIALS

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ABSTRACT. In this paper, we consider the degenerate Apostol-Bernoulli numbers and polynomials due to L. Carlitz. and we investigate some properties for these numbers and polynomials

1. Introduction

As is well known, the Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (\text{see [1–11]}). \quad (1.1)$$

When $x = 0$, $B_n = B_n(0)$, ($n \geq 0$) are called the Bernoulli numbers. From (1.1), we note that

$$(B+1)^n - B_n = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases} \quad \text{and } B_0 = 1, \quad (1.2)$$

with the usual convention about replacing B^n by B_n . It is not difficult to show that

$$B_n(x) = \sum_{l=0}^n \binom{n}{l} B_l x^{n-l}, \quad (\text{see [11]}). \quad (1.3)$$

It is well known that the Bernoulli polynomials of the second kind are given by the generating function to be

$$\frac{t}{\log(1+t)} (1+t)^x = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!}, \quad (\text{see [11]}). \quad (1.4)$$

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When $x = 0$, $b_n = b_n(0)$, are called the Bernoulli numbers of the second kind. In [4,5], L. Carlitz Introduced the degenerate Bernoulli polynomials which are given by the generating function to be

$$\frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}. \quad (1.5)$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$, are called the degenerate Bernoulli numbers. It is not difficult to show that

$$\lim_{\lambda \rightarrow 0} \beta_{n,\lambda} = B_n(x), \quad (n \geq 0). \quad (1.6)$$

From (1.5), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!} &= \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} \\ &= \left(\sum_{l=0}^{\infty} \beta_{l,\lambda} \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} \binom{\frac{x}{\lambda}}{m} \lambda^m t^m \right) \\ &= \left(\sum_{l=0}^{\infty} \beta_{l,\lambda} \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} (x|\lambda)_m \frac{t^m}{m!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} \beta_{l,\lambda} (x|\lambda)_{n-l} \right) \frac{t^n}{n!}, \end{aligned} \quad (1.7)$$

where $(x|\lambda)_0 = 1$, $(x|\lambda)_n = x(x - \lambda) \cdots (x - (n-1)\lambda)$, $(n \geq 1)$. Thus, by (1.7), we get

$$\beta_{n,\lambda}(x) = \sum_{l=0}^n \binom{n}{l} \beta_{l,\lambda} (x|\lambda)_{n-l}, \quad (n \geq 0). \quad (1.8)$$

By (1.5), we easily get

$$\sum_{a=0}^{d-1} (1 + \lambda t)^{\frac{a}{\lambda}} = \frac{1}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} \left((1 + \lambda t)^{\frac{d}{\lambda}} - 1 \right), \quad (d \in \mathbb{N}). \quad (1.9)$$

Thus, by (1.9), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\sum_{a=0}^{d-1} (a|\lambda)_n \right) \frac{t^n}{n!} &= \frac{1}{t} \left\{ \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{d}{\lambda}} - \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} \right\} \\ &= \sum_{n=0}^{\infty} \left\{ \frac{\beta_{n+1,\lambda}(d) - \beta_{n+1,\lambda}}{n+1} \right\} \frac{t^n}{n!}. \end{aligned} \quad (1.10)$$

From (1.10), we have

$$\sum_{a=0}^{d-1}(a|\lambda)_n = \frac{\beta_{n+1,\lambda}(d) - \beta_{n+1,\lambda}}{n+1}, \quad (1.11)$$

where $d \in \mathbb{N}$ and $n \geq 0$. For $w(\neq 1)$, the Apostol-Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{we^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,w}(x) \frac{t^n}{n!}, \quad (\text{see } [1-3, 6-10]). \quad (1.12)$$

When $x = 0$, $B_{n,w} = B_{n,w}(0)$ are called the Apostol-Bernoulli numbers. From (1.12), we note that

$$\begin{aligned} t &= (we^t - 1) \sum_{n=0}^{\infty} B_{n,w} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(w \sum_{l=0}^n \binom{n}{l} B_{l,w} - B_{n,w} \right) \frac{t^n}{n!}. \end{aligned} \quad (1.13)$$

By comparing the coefficients on the both sides of (1.13), we get

$$w \sum_{l=0}^n \binom{n}{l} B_{l,w} - B_{n,w} = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1. \end{cases} \quad (1.14)$$

From (1.14), we note that $B_{0,w} = 0 = B_{0,w}(x)$. In this paper, we consider the degenerate Apostol-Bernoulli numbers and polynomials due to L. Carlitz. and we investigate some properties for those numbers and polynomials.

2. Degenerate Apostol-Bernoulli polynomials

For $\lambda, w \in \mathbb{R}$ with $w \neq 1$, we consider the degenerate Apostol-Bernoulli polynomials which are defined by the generating function to be

$$\frac{t}{w(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda,w}(x) \frac{t^n}{n!}. \quad (2.1)$$

Note that $\lim_{\lambda \rightarrow 0} \beta_{n,\lambda,w}(x) = B_{n,w}(x)$, ($n \geq 0$). When $x = 0$, $\beta_{n,\lambda,w} = \beta_{n,\lambda,w}(0)$ are called the degenerate Apostol-Bernoulli numbers. From (2.1), we note that

$$\begin{aligned} t &= \left(\sum_{l=0}^{\infty} \beta_{l,\lambda,w} \frac{t^l}{l!} \right) \left(w(1+\lambda t)^{\frac{1}{\lambda}} - 1 \right) \\ &= \left(\sum_{l=0}^{\infty} \beta_{l,\lambda,w} \frac{t^l}{l!} \right) \left(w \sum_{m=0}^{\infty} (1|\lambda)_m \frac{t^m}{m!} - 1 \right) \\ &= \sum_{n=0}^{\infty} \left(w \sum_{l=0}^n \binom{n}{l} \beta_{l,\lambda,w} (1|\lambda)_{n-l} - \beta_{n,\lambda,w} \right) \frac{t^n}{n!}. \end{aligned} \quad (2.2)$$

Comparing the coefficients on the both sides of (2.2), we have

$$w \sum_{l=0}^n \binom{n}{l} \beta_{l,\lambda,w} (1|\lambda)_{n-l} - \beta_{n,\lambda,w} = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1. \end{cases} \quad (2.3)$$

By (2.3), we easily get

$$w\beta_{0,\lambda,w} - \beta_{0,\lambda,w} = 0 \rightarrow \beta_{0,\lambda,w} = \frac{0}{w-1} = 0. \quad (2.4)$$

For $n = 1$, we have

$$1 = w \sum_{l=0}^1 \binom{1}{l} \beta_{l,\lambda,w} (1|\lambda)_{1-l} - \beta_{1,\lambda,w} = w\beta_{1,\lambda,w} - \beta_{1,\lambda,w}. \quad (2.5)$$

Thus, by (2.5), we get $\beta_{1,\lambda,w} = \frac{1}{w-1}$. Let $n = 2$. Then, by (2.3), we get

$$\begin{aligned} 0 &= w \sum_{l=0}^2 \binom{2}{l} \beta_{l,\lambda,w} (1|\lambda)_{2-l} - \beta_{2,\lambda,w} \\ &= 2w\beta_{1,\lambda,w} (1|\lambda)_1 + w\beta_{2,\lambda,w} - \beta_{2,\lambda,w} \\ &= \frac{2w}{w-1} + (w-1)\beta_{2,\lambda,w} \end{aligned} \quad (2.6)$$

From (2.6), we have

$$\beta_{2,\lambda,w} = -\frac{2w}{(w-1)^2}. \quad (2.7)$$

For $n = 3$, we have

$$\begin{aligned} 0 &= w \sum_{l=0}^3 \binom{3}{l} \beta_{l,\lambda,w}(1|\lambda)_{3-l} - \beta_{3,\lambda,w} \\ &= 3w\beta_{1,\lambda,w}(1|\lambda)_2 + 3w\beta_{2,\lambda,w}(1|\lambda)_1 + 3\beta_{3,\lambda,w} - \beta_{3,\lambda,w} \\ &= -\frac{3w}{w-1}(1-\lambda) + \frac{6w^2}{(w-1)^2} + (w-1)\beta_{3,\lambda,w}. \end{aligned} \quad (2.8)$$

By (2.8), we get

$$\beta_{3,\lambda,w} = \frac{3w}{(w-1)^2}(1-\lambda) - \frac{6w^2}{(w-1)^3}, \dots \quad (2.9)$$

Now, we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} \beta_{n,\lambda,w}(x) \frac{t^n}{n!} &= \frac{t}{w(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} \\ &= \left(\sum_{l=0}^{\infty} \beta_{l,\lambda,w} \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} \binom{\frac{x}{\lambda}}{m} \lambda^m t^m \right) \\ &= \left(\sum_{l=0}^{\infty} \beta_{l,\lambda,w} \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} (x|\lambda)_m \frac{t^m}{m!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} \beta_{l,\lambda,w}(x|\lambda)_{n-l} \right) \frac{t^n}{n!}. \end{aligned} \quad (2.10)$$

By comparing the coefficients on the both sides of (2.10), we get

$$\beta_{n,\lambda,w}(x) = \sum_{l=0}^n \binom{n}{l} \beta_{l,\lambda,w}(x|\lambda)_{n-l}, \quad (n \geq 0). \quad (2.11)$$

Note that

$$\lim_{\lambda \rightarrow 0} \beta_{n,\lambda,w}(x) = \sum_{l=0}^n \binom{n}{l} B_{l,w} x^{n-l} = B_{n,w}(x), \quad (n \geq 0).$$

From (2.1), we note that

$$\begin{aligned}
& \sum_{n=0}^{\infty} \{w\beta_{n,\lambda,w}(x+1) - \beta_{n,\lambda,w}(x)\} \frac{t^n}{n!} \\
&= w \frac{t}{w(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x+1}{\lambda}} - \frac{t}{w(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} \\
&= \frac{t}{w(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} \left(w(1+\lambda t)^{\frac{1}{\lambda}} - 1 \right) \\
&= t(1+\lambda t)^{\frac{x}{\lambda}} = t \sum_{n=0}^{\infty} (x|\lambda)_n \frac{t^n}{n!}.
\end{aligned} \tag{2.12}$$

Since $\beta_{0,\lambda,w} = \beta_{0,\lambda,w}(x) = 0$. By (2.12), we get

$$\begin{aligned}
& \sum_{n=0}^{\infty} (x|\lambda)_n \frac{t^n}{n!} = \frac{1}{t} \sum_{n=0}^{\infty} \{w\beta_{n,\lambda,w}(x+1) - \beta_{n,\lambda,w}(x)\} \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left\{ \frac{w\beta_{n+1,\lambda,w}(x+1) - \beta_{n+1,\lambda,w}(x)}{n+1} \right\} \frac{t^n}{n!}.
\end{aligned} \tag{2.13}$$

Comparing the coefficients on the both sides of (2.13), we have

$$(x|\lambda)_n = \frac{1}{n+1} (w\beta_{n+1,\lambda,w}(x+1) - \beta_{n+1,\lambda,w}(x)), \quad (n \geq 0). \tag{2.14}$$

Now, we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} \beta_{n,-\lambda,w^{-1}}(-x) \frac{t^n}{n!} &= \frac{t}{w^{-1}(1-\lambda t)^{-\frac{1}{\lambda}} - 1} (1-\lambda t)^{\frac{x}{\lambda}} \\
&= w \frac{t}{1-w(1-\lambda t)^{\frac{1}{\lambda}}} (1-\lambda t)^{\frac{x+1}{\lambda}} \\
&= w \frac{(-t)}{w(1-\lambda t)^{\frac{1}{\lambda}} - 1} (1-\lambda t)^{\frac{x+1}{\lambda}} \\
&= w \sum_{n=0}^{\infty} \beta_{n,\lambda,w}(x+1)(-1)^n \frac{t^n}{n!}.
\end{aligned} \tag{2.15}$$

By comparing the coefficients on the both sides of (2.15), we obtain the following equation:

$$(-1)^n \beta_{n,-\lambda,w^{-1}}(-x) = w\beta_{n,\lambda,w}(x+1), \quad (n \geq 0). \tag{2.16}$$

For $d \in \mathbb{N}$, we have

$$\begin{aligned}
\frac{t}{w(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} &= \frac{t}{w^d(1+\lambda t)^{\frac{d}{\lambda}} - 1} \sum_{i=0}^{d-1} w^i (1 + \lambda t)^{\frac{i+x}{\lambda}} \\
&= \sum_{i=0}^{d-1} w^i \frac{t}{w^d(1+\lambda t)^{\frac{d}{\lambda}} - 1} (1 + \lambda t)^{\frac{i+x}{\lambda}} \\
&= \frac{1}{d} \sum_{i=0}^{d-1} w^i \frac{dt}{w^d(1+\frac{\lambda}{d}dt)^{\frac{d}{\lambda}} - 1} (1 + \frac{\lambda}{d}dt)^{\frac{d}{\lambda} \frac{i+x}{d}} \\
&= \sum_{n=0}^{\infty} \left(d^{n-1} \sum_{i=0}^{d-1} w^i \beta_{n, \frac{\lambda}{d}, w^d} \left(\frac{x+i}{d} \right) \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.17}$$

From (2.1) and (1.13), we can derive the following equations (2.18):

$$\beta_{n, \lambda, w}(x) = d^{n-1} \sum_{i=0}^{d-1} w^i \beta_{n, \frac{\lambda}{d}, w^d} \left(\frac{x+i}{d} \right), \quad (d \in \mathbb{N}, n \geq 0). \tag{2.18}$$

By (2.1), we get

$$\begin{aligned}
\sum_{a=0}^{d-1} (1 + \lambda t)^{\frac{a}{\lambda}} w^a &= \frac{1}{w(1 + \lambda t)^{\frac{1}{\lambda}} - 1} \left((1 + \lambda t)^{\frac{d}{\lambda}} w^d - 1 \right) \\
&= \frac{1}{t} \left\{ w^d \frac{t}{w(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{d}{\lambda}} - \frac{t}{w(1 + \lambda t)^{\frac{1}{\lambda}} - 1} \right\} \\
&= \frac{1}{t} \left\{ \sum_{n=0}^{\infty} (w^d \beta_{n, \lambda, w}(d) - \beta_{n, \lambda, w}) \frac{t^n}{n!} \right\} \\
&= \sum_{n=0}^{\infty} \left\{ \frac{w^d \beta_{n+1, \lambda, w}(d) - \beta_{n+1, \lambda, w}}{n+1} \right\} \frac{t^n}{n!}.
\end{aligned} \tag{2.19}$$

It is not difficult to show that

$$\begin{aligned}
\sum_{a=0}^{d-1} (1 + \lambda t)^{\frac{a}{\lambda}} w^a &= \sum_{a=0}^{d-1} w^a \sum_{n=0}^{\infty} \binom{\frac{a}{\lambda}}{n} \lambda^n t^n \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{a=0}^{d-1} (a|\lambda)_n w^a \right\} \frac{t^n}{n!}.
\end{aligned} \tag{2.20}$$

Therefore, by (2.19) and (2.20), we obtain the following equation (2.21).

$$\sum_{a=0}^{d-1} (a|\lambda)_n w^a = \frac{1}{n+1} \left\{ w^d \beta_{n+1, \lambda, w}(d) - \beta_{n+1, \lambda, w} \right\}, \tag{2.21}$$

where $n \geq 0$ and $d \in \mathbb{N}$. For $n \geq 0$, the stirling number of the first kind is defined as

$$(x)_n = \sum_{l=0}^n S_1(n, l)x^l, \quad (2.22)$$

and the stirling number of the second kind is defined by

$$x^n = \sum_{l=0}^n S_2(n, l)(x)_l, \quad (2.23)$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, ($n \geq 1$). By replacing t by $\frac{1}{\lambda}(e^{\lambda t} - 1)$ in (2.1), we get

$$\begin{aligned} & \sum_{m=0}^{\infty} \beta_{m,\lambda,w}(x) \frac{1}{m!} \lambda^{-m} (e^{\lambda t} - 1)^m = \frac{\frac{1}{\lambda}(e^{\lambda t} - 1)}{we^t - 1} e^{xt} \\ &= \frac{1}{\lambda} \frac{1}{we^t - 1} (e^{(\lambda+x)t} - e^{xt}) = \frac{1}{\lambda t} \left(\frac{t}{we^t - 1} e^{(\lambda+x)t} - \frac{t}{we^t - 1} e^{xt} \right) \\ &= \frac{1}{\lambda t} \sum_{n=1}^{\infty} (B_{n,w}(x+\lambda) - B_{n,w}(x)) \frac{t^n}{n!} = \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{B_{n+1,w}(x+\lambda) - B_{n+1,w}(x)}{n+1} \right) \frac{t^n}{n!}. \end{aligned} \quad (2.24)$$

Note that

$$\begin{aligned} & \sum_{m=0}^{\infty} \beta_{m,\lambda,w}(x) \lambda^{-m} \frac{1}{m!} (e^{\lambda t} - 1)^m = \sum_{m=0}^{\infty} \beta_{m,\lambda,w}(x) \lambda^{-m} \sum_{n=m}^{\infty} S_2(n, m) \lambda^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \lambda^{n-m} S_2(n, m) \beta_{m,\lambda,w}(x) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.25)$$

Thus, by (2.24) and (2.25), we get

$$\frac{1}{n+1} (B_{n+1,w}(x+\lambda) - B_{n+1,w}(x)) = \sum_{m=0}^n \lambda^{n-m+1} S_2(n, m) \beta_{m,\lambda,w}(x), \quad (2.26)$$

where $n \geq 0$. From (2.1), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \beta_{n,\lambda,w}(x) \frac{t^n}{n!} &= \frac{t}{w(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} \\
&= \frac{t}{we^{\frac{1}{\lambda} \log(1+\lambda t)} - 1} e^{\frac{x}{\lambda} \log(1+\lambda t)} \\
&= \left(\frac{\lambda t}{\log(1+\lambda t)} \right) \left(\frac{\frac{1}{\lambda} \log(1+\lambda t)}{we^{\frac{1}{\lambda} \log(1+\lambda t)} - 1} \right) e^{\frac{x}{\lambda} \log(1+\lambda t)} \\
&= \left(\sum_{k=0}^{\infty} b_k \lambda^k \frac{t^k}{k!} \right) \left(\sum_{l=0}^{\infty} B_{l,w}(x) \frac{1}{l!} \lambda^{-l} (\log(1+\lambda t))^l \right) \\
&= \left(\sum_{k=0}^{\infty} b_k \lambda^k \frac{t^k}{k!} \right) \left(\sum_{l=0}^{\infty} B_{l,w}(x) \lambda^{-l} \sum_{m=l}^{\infty} S_1(m, l) \frac{\lambda^m}{m!} t^m \right) \tag{2.27} \\
&= \left(\sum_{k=0}^{\infty} b_k \lambda^k \frac{t^k}{k!} \right) \left(\sum_{m=0}^{\infty} \left(\sum_{l=0}^m B_{l,w}(x) \lambda^{m-l} S_1(m, l) \right) \frac{t^m}{m!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \sum_{l=0}^m B_{l,w}(x) \lambda^{m-l} S_1(m, l) b_{n-m} \lambda^{n-m} \binom{n}{m} \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \sum_{l=0}^m \binom{n}{m} \lambda^{n-l} B_{l,w}(x) S_1(m, l) b_{n-m} \right) \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients on the both sides of (2.27), we have

$$\beta_{n,\lambda,w}(x) = \sum_{m=0}^n \sum_{l=0}^m \binom{n}{m} \lambda^{n-l} B_{l,w}(x) S_1(m, l) b_{n-m}, \tag{2.28}$$

where $n \geq 0$.

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